III. MATHEMATICAL TOOLS

Digitisation and resolution of linear differential equations

III.1 Digitisation of variables

N.B.:

This tool, used in physics, employs certain mathematical concepts in terms of its internal logic.

III.1.1 Independent Variables

General remarks :

Certain authors (non-mathematicians) confuse discontinuity with the digital, function with distribution and even a set and a series. In order to make things clear :

- *Ratio* refers to a function or distribution.
- Only the family of *applications*(1) of ratios is of interest here.
- The following definitions are applied here :
  - *Digital variable*, the monotonous set of $\mathbb{Z}$;
  - *Digital function*, a function containing digital variables ;
  - *Analogical function*, any application that uses continuous variables (admitting dx).

It should be mentioned in passing that punctual ratios such as the function $\delta(x)$ or the set $\delta(x - pX)$ of DIRAC are analogic(2).

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(1) As a reminder, an application is a surjective left-handed and univocal ratio $(x \mapsto g)$.

(2) $p \in \mathbb{Z}$ (to simplify) and X is the period. According to L. SCHWARTZ, $\delta(x)$ is a function but its derivatives are distributions.

Concerning discontinuity :

Normally, discontinuity is an epithet used to describe ratios that are non-derivative in certain respects. A discontinuity such as this poses no problem for digitisation. To be more specific, discontinuity is a vertical fracture that, at one point in the variable, produces two different values for the ratio. This type of discontinuity is the subject of distributions(1), at least for physicists. For this purpose, the digitisation process needs to be adapted slightly.

Concerning the variation :

Let there be a ratio g(x) between two variables : x being independent and with uniform increase (at constant variation) and g being dependent on x. This ratio translates *the variation* of g in relation to x in three forms : analytic, graphic or digital. When g(x) is analytical and indefinitely derivable, $\frac{dg}{dx}$ is used to designate *the rate* or *speed* of this variation, through $\frac{d^2g}{dx^2}$ *the rate of the rate* of variation and so on. It should be pointed out immediately that the significant interest of a FOURIER transform is in the transcription of a differential into an algebraic term.

Dyadic spaces :

It has been established that the binary numbers 0 and 1 are the simplest, most intuitive and most natural numbers conceivable. These two fundamental states(2) of understanding constitute the basis for any digital mathematical construction and formal logic.

A number containing N binary figures represents 2^N different states. In dyadic terms, a space having N dimensions comprises 2^N points (or positions). These positions, ordered according to their decimal values, constitute a preponderant set in digital processing.

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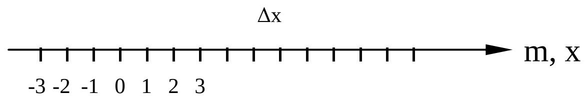
(1) Do not confuse the discontinuity of distribution with the continuity of their variables.

(2) The statement and its negation such as "yes" or "no", "true" or "false", etc …

Digital variation :

The variable $m \in \mathbb{Z}$, discrete by definition, does not accept analytical operators such as derivation and integration. It is consequently excluded from equations in physics. To overcome this difficulty, m is associated with the continuous axis of x in the following manner :



(Fig. III.1a)

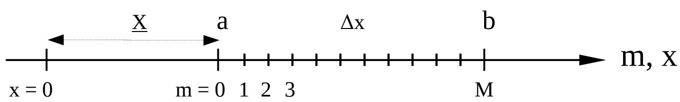
where Δx mathematically represents the unit. It is the passage from "discrete digitisation" to "analytic digitisation" dedicated to the substitution $m \leftarrow m\Delta x$. On the other hand, the switch from continuous to digital via $x \leftarrow m\Delta x$ is known as "digitisation". It assigns to Δx the relationship of scale between x and m. Finally, the term "associated digital variable " or more simply[1] "digital variable" is assigned to the concept $m\Delta x$.

Variation interval[2]**:**

Digital calculation uses finite values of variables and functions. This terminates the variation interval as follows :

$$x \in [a, b] \ ; \ b - a \ \underline{\Delta} \ X$$

It is always desirable in this case to restrict digitisation to this interval and, if possible, to make the origins of x and m coincide. Where these origins are fatally separate, the substitution $x \leftarrow \underline{X} + m\Delta x$ takes their place :



(Fig. III.1b)

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(1) without possible risk of confusion.
(2) It corresponds physically with the field of observation, action or measurement.

Note that the property of periodicity (see A.6.2) of FOURIER digital transforms make it possible to begin the intervals of x and f at the origin of these coordinates. The result is that $\underline{X}$, like $\underline{F}$ (the equivalent of $\underline{X}$ over f) are virtually nil.

Sampling :

If digitisation is a sort of change of variable (x 🡐 mΔx ; Δx being the indivisible step of the measurement) applied unrestrictedly to the axis of x, sampling is a technique to be applied to the interval X, one that should satisfy the dyadic requirement :

$$M = 2^N - 1 \quad ; \quad M \triangleq \left.\frac{X}{\Delta x}\right|_{int}$$

Furthermore, if M is too large, the choice of Δx must maintain the variation of the function being processed, with a single Δx that is less than the measurement threshold.

III.1.2 Dependent variables (functions)

Description :

Definition :

Like the digital variable, the standard application g(mΔx). is known as the "associated digital function" or simply the "digital function".

Criteria :

1) The analogical function and its digital version share the same analytical properties.
2) The digital function must be locally integrable(1).
3) Unlike the variable, the digital function may take any finite value.

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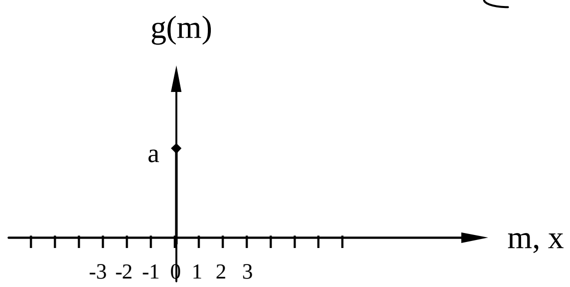
(1) Suitably digitised distributions are also integrable within the meaning attributed to them by RIEMANN.

Comment :

All analogical ratios originating in physics can be converted into digitals.

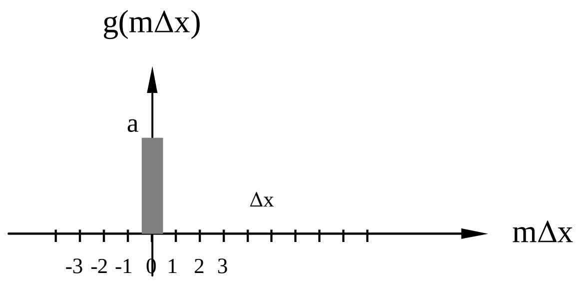
Integration :

This is the function defined by : $g(m) = \begin{cases} a & \text{for } m = 0 \,; \\ 0 & \text{elsewhere.} \end{cases}$



(Fig. III.2a)

It is obvious that the Riemannian integration of this function over x is nil. This means that g(m) is independent of x and devoid of analytical meaning. In order to appropriate g(m) for integration, one must resort to the substitution $m \leftarrow m\Delta x$ so that it produces :



(Fig. III.2b)

$$\int_{-\infty}^{\infty} g(m\Delta x)\, dx \; = \; a \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} dx \; = \; a\, \Delta x \qquad \text{(III.1)}$$

Functions of DIRAC :

Analogical expression :

It is normal to use the term DIRAC "impulsion" (or measurement) to describe the function :

$$\delta(x) = \begin{cases} 1 & \text{for } x = 0 \,; \\ 0 & \text{elsewhere.} \end{cases}$$

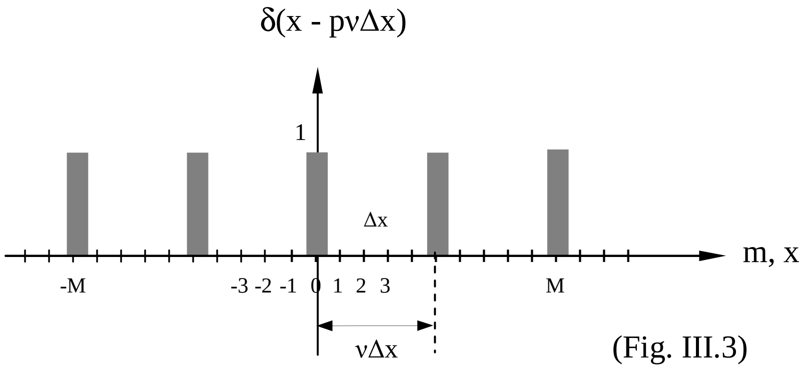
The uniform repetition (at constant interval) of this measurement generates the set :

$$\delta(x - pX) = \begin{cases} 1 & \text{for } x = pX \;;\; (p = 0, \pm 1, \pm 2, \ldots, \pm\infty) \\ 0 & \text{elsewhere.} \end{cases}$$

This set, as well as the impulse, are devoid of Riemannian integration.

Digital version :

The digitisation of the x axis will verify the equality of : $X = \nu\Delta x$; ν being a whole number.



(Fig. III.3)

Integration :

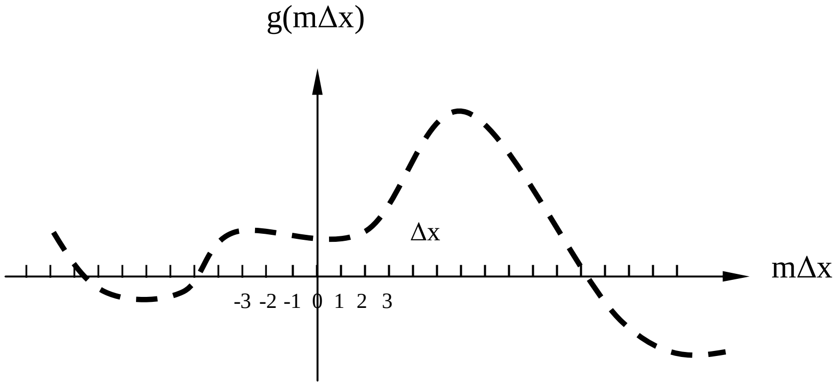
The previous set is only integrable on an interval terminating in x, [-M, M] for example, the result of the formula (III.1) is :

$$\int_{-M\Delta x}^{M\Delta x} \delta(x - p\nu\Delta x)\, dx = \sum_{p=-\frac{M}{\nu}}^{\frac{M}{\nu}} \delta(x - p\nu\Delta x)\, \Delta x = \left(1 + 2\frac{M}{\nu}\right) \Delta x$$

Typical functions :

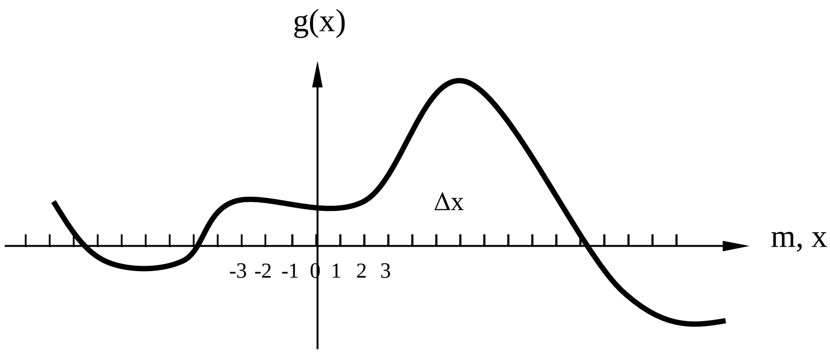
Mono-variable function :

Let there be the function :



(Fig. III.4a)

The substitution $m\Delta x \leftarrow x$ incontinently restores the analogical version :



(Fig. III.4b)

$g(m\Delta x)$ is commonly expressed as follows :

$$g(m\Delta x) \triangleq g(x)\,\delta(x - m\Delta x) \;\; ; \;\; (m = 0, \pm 1, \pm 2, \ldots, \pm\infty) \qquad \text{(III.2a)}$$

with : $\delta(x - m\Delta x) \triangleq \begin{cases} 1 & \text{for } x = m\Delta x \;\; ; \\ 0 & \text{elsewhere.} \end{cases}$

Integration :

Assuming that $g(x)$ is integrable, this produces :

$$\int_{-\infty}^{\infty} g(m\Delta x)\,dx = \int_{-\infty}^{\infty} g(x)\,\delta(x - m\Delta x)\,dx = \Delta x \sum_{m=-\infty}^{\infty} g(m\Delta x)$$

Multivariable function*[1]*:

Let us consider the function :

$$g(m_1\Delta x_1, \ldots, m_N\Delta x_N) \triangleq g(\vec{x})\,\delta(x_1 - m_1\Delta x_1)\ldots\delta(x_N - m_N\Delta x_N)$$

$$(m_1, m_2, \ldots, m_N = 0, \pm 1, \pm 2, \ldots, \pm\infty) \qquad \text{(III.2b)}$$

Integration :

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(\vec{x})\,\delta(x_1 - m_1\Delta x_1)\ldots\delta(x_N - m_N\Delta x_N)\,dx_1 dx_2 \ldots dx_N$$

$$= \Delta x_1 \ldots \Delta x_N \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_N=-\infty}^{\infty} g(m_1\Delta x_1, \ldots, m_N\Delta x_N)$$

Convolution :

Analogical expression :

The product of the convolution of two mono-variable functions is written as :

$$g_1(x) * g_2(x) \triangleq \int_{-\infty}^{\infty} g_1(\xi)\,g_2(x-\xi)\,d\xi$$

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(1) It is convenient to transcribe the multivariable functions into multi-dimensional spaces.

In the case of a multivariable function, it is written :

$$g_1(\vec{x})*g_2(\vec{x}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g_1(\vec{\xi})\, g_2(\vec{x}-\vec{\xi})\, d\xi_1 \dots d\xi_N$$

Digital version :

Along the same lines of development, the following can be deduced :

$$g_1(m\Delta x)*g_2(m\Delta x) = \Delta x \sum_{k=-\infty}^{\infty} g_1(k\Delta x)\, g_2\big[(m-k)\Delta x\big] \;; \quad (m = 0, \pm 1, \pm 2, \dots, \pm\infty) \qquad \text{(III.3a)}$$

$$g_1(m_1\Delta x_1, \dots, m_N\Delta x_N)*g_2(m_1\Delta x_1, \dots, m_N\Delta x_N)$$
$$= \Delta x_1 \dots \Delta x_N \sum_{k_1=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} g_1(k_1\Delta x_1, \dots, k_N\Delta x_N) \times$$
$$g_2\big[(m_1-k_1)\Delta x_1, \dots, (m_N-k_N)\Delta x_N\big] \;;$$
$$(m_1, m_2, \dots, m_N = 0, \pm 1, \pm 2, \dots, \pm\infty) \qquad \text{(III.3b)}$$

Periodic functions :

Analogical expression :

A mono-variable periodic function is expressed :

$$g(x) \underline{\Delta}\ g(x - pX) = g_1(x)*\delta(x - pX) \;; \; (p = 0, \pm 1, \pm 2, \dots, \pm\infty) \qquad \text{(III.4a)}$$

where $g_1(x)$ is the main period defined on $x \in [0, X]$.

Comment :

A periodic function is continuous if $g_1(0) = g_1(X)$ and discontinuous(1) in the opposite case.

Where the periodicity is multivariable, the result is :

$$g(\vec{x}) = g(x_1 - p_1X_1, \dots, x_N - p_NX_N)$$
$$= g_1(\vec{x})*\delta(x_1 - p_1X_1)* \dots *\delta(x_N - X_N) \;;$$
$$(p_1, p_2 \dots, p_N = 0, \pm 1, \pm 2, \dots, \pm\infty) \qquad \text{(III.4b)}$$

(1) It is possible for $g_1(X)$ itself to be discontinuous (a distribution).

Digital version :

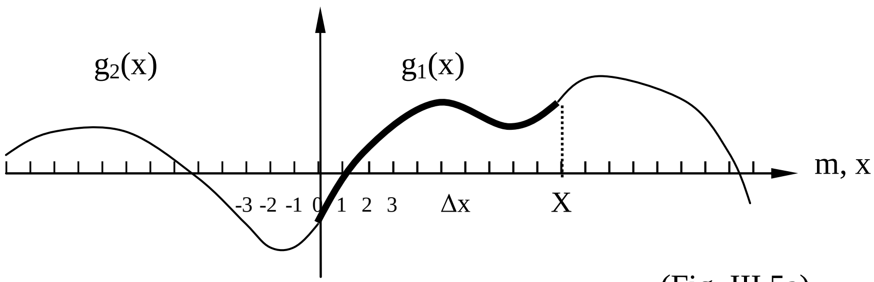
Since the periodic function is defined by its main period, sampling is limited to this period. As a result :

$$g_1(m\Delta x) = g_1(x)\,\delta(x - m\Delta x) \;;\; (m = 0, 1, 2, \ldots, M) \;;$$
$$M\Delta x = X \qquad \text{(III.5a)}$$

$$g_1(m_1\Delta x_1, \ldots, m_N\Delta x_N) = g_1(\vec{x})\,\delta(x_1 - m_1\Delta x_1) \ldots \delta(x_N - m_N\Delta x_N)$$
$$(m_1 = 0, 1, 2, \ldots, M_1 \;; \ldots ;\; m_N = 0, 1, 2, \ldots, M_N) \qquad \text{(III.5b)}$$

Aperiodic functions :

Let the function $g_2(x)$ be continuous and locally integrable :



(Fig. III.5a)

Let us only consider the part[1] $g_1(x)$ over $x \in [0, X]$ of $g_2(x)$. This can be tackled in two ways :

1) By replacing $g_2(x)$ with the above periodic function[2]:

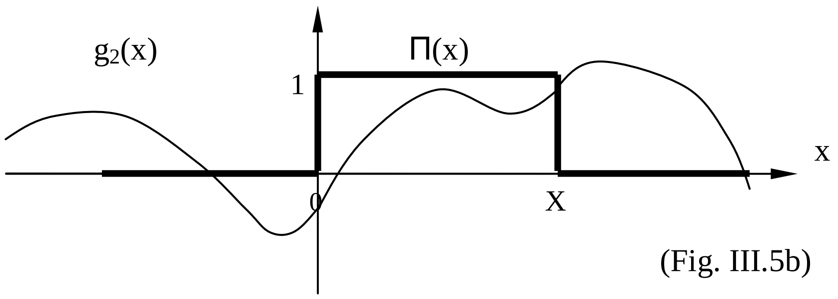
$$g(x) = g_1(x) * \delta(x - pX) \;;\; (p = 0, \pm 1, \pm 2, \ldots, \pm\infty)$$

which can be digitised in accordance with (III.5a).

(1) It recalls the field of action mentioned above.
(2) This is justified by the fact that $g(x)$ reproduces $g_1(x)$ all along the axis x.

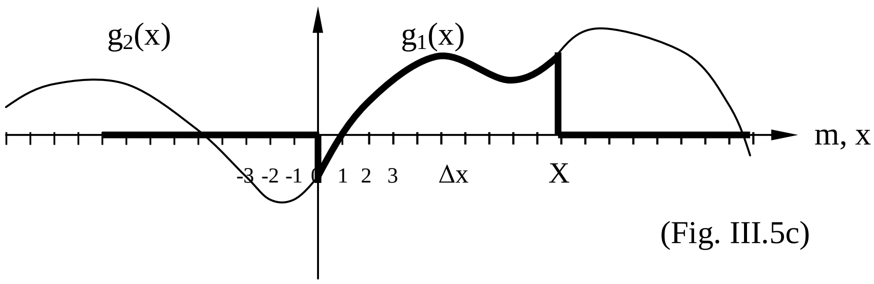
At any event, the property of (A.6.2) converts any digitised function into a periodic function having the interval of variation as the period.

2) If we multiply $g_2(x)$ by the function $\Pi(x)$:



(Fig. III.5b)

this immediately produces : $g_1(x) = \Pi(x) \times g_2(x)$



(Fig. III.5c)

Derivatives :

The following formulae are shown respectively in (A.6.1) and (A.6.2) :

$$g^{(n)}(x) = g(x) * \delta^{(n)}(x) \qquad \text{(III.6a)}$$

$$g^{(n)}(m\Delta x) = g(m\Delta x) * \delta^{(n)}(x) \;\; ; \;\; (m = 0, \pm 1, \pm 2, \ldots, \pm\infty) \qquad \text{(III.6b)}$$

III.2 Resolution of the equations

To simplify this essay, we shall restrict ourselves to mono-variable equations.

III.2.1 Algebraic equations

Equation lacking a second member :

Consider the equation :

$$g(x)(x - \xi_1)(x - \xi_2)\ldots(x - \xi_K) \;=\; 0 \;\;;\;\; g(x) \neq 0 \qquad \text{(III.7a)}$$

This permits the following solutions :

$$g(x) \;=\; a_1\,\delta(x - \xi_1),\; a_2\,\delta(x - \xi_2),\; \ldots,\; a_K\,\delta(x - \xi_K) \qquad \text{(III.7b)}$$

in which a_1, a_2, …, a_K are arbitrary constants in the absence of the initial conditions. The latter must correspond to points ξ_1, ξ_2, … and be limited in number to K.

In digital form, the arbitrary constants are predictable. They are assigned the default unit where the initial conditions are lacking. In this order, the equation (III.7b) is written :

$$g(m\Delta x) \;=\; \delta(x - m_1\Delta x),\; \delta(x - m_2\Delta x),\; \ldots,\; \delta(x - m_K\Delta x) \qquad \text{(III.7c)}$$

Two-member equation :

Let : $g(x)(x - \xi_1)(x - \xi_2)\ldots(x - \xi_K) \;=\; p(x)$ (III.8a)

It can be seen that where $x = \xi_1$ or $x = \xi_2$ …, the equation loses its second member :

$$p(x) \leftarrow 0 \;\text{ and }\; g(x) \;=\; a_1\,\delta(x - \xi_1),\; a_2\,\delta(x - \xi_2),\; \ldots,\; a_K\,\delta(x - \xi_K)$$

Outside the points ξ_1, ξ_2 , …, this produces :

$$g(x) \;=\; \frac{p(x)}{(x - \xi_1)(x - \xi_2)\ldots(x - \xi_K)}$$

In the same way, the general solution of the equation (III.8a) produces the following result :

$$g(x) = a_1\delta(x - \xi_1), a_2\delta(x - \xi_2), \ldots, a_K\delta(x - \xi_K), \frac{p(x)}{(x-\xi_1)(x-\xi_2)\ldots(x-\xi_K)} \qquad \text{(III.8b)}$$

The digital version of this equation is written :

$$g(m\Delta x) = \delta(x - m_1\Delta x), \ldots, \delta(x - m_K\Delta x), \frac{p(m\Delta x)}{(x-m_1\Delta x)\ldots(x-m_K\Delta x)} \qquad \text{(III.8c)}$$

III.2.2 Differential equations

General form :

A linear differential equation is expressed as :

$$p_n(x)\frac{d^n g(x)}{dx^n} + p_{n-1}(x)\frac{d^{n-1} g(x)}{dx^{n-1}} + \ldots + p_1(x)\frac{dg(x)}{dx} + p_o(x)\, g(x) = p(x) \qquad \text{(III.9)}$$

where $p_n(x), \ldots, p_o(x)$ and $p(x)$ are known functions. It is fascinating to recall that the origin of this equation lies in physics, all of the components being locally integrable.

The digitisation of (III.9) results from the substitution :

$$x \Leftarrow m\Delta x \quad ; \quad (m = 0, \pm 1, \pm 2, \ldots, \pm\infty)$$

Resolution schematic :

The equation (III.9) can be solved in four stages :

1) Digitising the equation by determining Δx and M ;
2) Applying TF (see A.6) to all its members ;
3) Separating the knowns from the unknowns ;
4) Proceeding to TF^{-1}.

Example :

For the sake of simplicity, let us make do with a 2nd degree equation :

$$p_2(x)\,\frac{d^2g(x)}{dx^2} + p_1(x)\,\frac{dg(x)}{dx} + p_o(x)\,g(x) = p(x) \qquad \text{(III.10a)}$$

The digitisation thereof for a terminated interval produces :

$$p_2(m\Delta x)\,\frac{d^2g(m\Delta x)}{dx^2} + p_1(m\Delta x)\,\frac{dg(m\Delta x)}{dx} +$$
$$p_o(m\Delta x)\,g(m\Delta x) = p(m\Delta x) \;\; ; \;\; (m = 0, 1, 2, \ldots, M)$$

The application TF produces :

$$\mathbf{p}_2(w\Delta f)*\left[(j2\pi w\Delta f)^2\,\mathbf{g}(w\Delta f)\right] + \mathbf{p}_1(w\Delta f)*\left[(j2\pi w\Delta f)\,\mathbf{g}(w\Delta f)\right] +$$
$$\mathbf{p}_o(w\Delta f)*\mathbf{g}(w\Delta f) = \mathbf{p}(w\Delta f) \;\; ; \;\; (w = 0, 1, 2, \ldots, W) \;\Rightarrow$$

$$\Delta f \sum_{k=0}^{W} \mathbf{g}[(w-k)\Delta f]\left\{\mathbf{p}_2(k\Delta f)\,[j2\pi(w-k)\Delta f]^2 + \mathbf{p}_1(k\Delta f)\,[j2\pi(w-k)\Delta f]\right.$$
$$\left. + \mathbf{p}_o(k\Delta f)\right\} = \mathbf{p}(w\Delta f) \;\; ; \;\; (w = 0, 1, 2, \ldots, W) \qquad \text{(III.10b)}$$

The periodicity of $\mathbf{g}(w)$ in W enables the substitution :

$$\mathbf{g}[(w-k)\Delta f] \;\Leftarrow\; \mathbf{g}[(W+w-k)\Delta f] \quad \text{where } w < k$$

As a result, the variation of w in the equation (III.10b) generates an algebraic system, from W + 1 linear equations, containing unknowns(1) W of the type $\mathbf{g}(w\Delta f)$.

The resolution of the latter results in the functions $\mathbf{g}(w\Delta f)$ which with the help of TF^{-1}, serve to reconstruct the function $g(m\Delta x)$.

Example of calculation :

$$\text{Let } \Delta f\left\{\mathbf{p}_2(k\Delta f)\,[j2\pi(w-k)\Delta f]^2 + \mathbf{P}_1(k\Delta f)\,[j2\pi(w-k)\Delta f] + \right.$$
$$\left. \mathbf{p}_o(k\Delta f)\right\} = \boldsymbol{\phi}(w\Delta f, k\Delta f)$$

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(1) Because the continuity of $\mathbf{g}(f)$ implies : $\mathbf{g}(0) = \mathbf{g}(W)$.

The following is deduced for :

<u>w = 0 :</u>

$$\mathbf{g}(0)\,\phi(0, 0) + \mathbf{g}[(W - 1)\Delta f]\,\phi(0, \Delta f) + \ldots + \mathbf{g}(0)\,\phi(0, W\Delta f) = \mathbf{p}(0)$$

<u>w = 1 :</u>

$$\mathbf{g}(\Delta f)\,\phi(\Delta f, 0) + \mathbf{g}(0)\,\phi(\Delta f, \Delta f) + \ldots + \mathbf{g}(\Delta f)\,\phi(\Delta f, W\Delta f) = \mathbf{p}(\Delta f)$$

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<u>w = W :</u>

$$\mathbf{g}(W\Delta f)\,\phi(W\Delta f, 0) + \mathbf{g}[(W - 1)\Delta f]\,\phi(W\Delta f, \Delta f) + \ldots + \mathbf{g}(0)\,\phi(W\Delta f, W\Delta f) = \mathbf{p}(W\Delta f)$$

Remembering that $\mathbf{g}(W\Delta f) = \mathbf{g}(0)$ and $\mathbf{p}(W\Delta f) = \mathbf{p}(0)$.

This system is finally shown as :

$$\left.\begin{array}{l}
c_{00}\,\mathbf{g}(0) + c_{01}\,\mathbf{g}(\Delta f) + \ldots + c_{0(W-1)}\,\mathbf{g}[(W - 1)\Delta f] = \mathbf{p}(0) \\
c_{10}\,\mathbf{g}(0) + c_{11}\,\mathbf{g}(\Delta f) + \ldots + c_{1(W-1)}\,\mathbf{g}[(W - 1)\Delta f] = \mathbf{p}(\Delta f) \\
\quad . \\
\quad . \\
\quad . \\
c_{(W-1)0}\,\mathbf{g}(0) + c_{(W-1)1}\,\mathbf{g}(\Delta f) + \ldots + \\
\qquad c_{(W-1)(W-1)}\,\mathbf{g}[(W - 1)\Delta f] = \mathbf{p}[(W - 1)\Delta f] \\
c_{W0}\,\mathbf{g}(0) + c_{W1}\,\mathbf{g}(\Delta f) + \ldots + c_{WW}\,\mathbf{g}[(W - 1)\Delta f] = \mathbf{p}(0)
\end{array}\right\} \quad \text{(III.10c)}$$

in which c_{wi} are numbers.

Special case :

For $p(m\Delta x) = 0$, the system (III.10c) accepts non-trivial solutions since two, and only two, of its equations (the first and last) are linearly dependent.